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Approximate Methods for Calculating the Properties of Heated Laminar Boundary Layers in Water

G. M. Harpole, S. A. Berger, J. Aroesty

A Report prepared for

DEFENSE ADVANCED RESEARCH PROJECTS AGENCY

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A laminar boundary layer integral method (the method of Thwaites and Walz) is extended to handle variable fluid properties. This integral method uses simple correlations of universal parameters based on similar boundary layer flows. These universal parameters are correlated for water, which has a strongly temperature dependent viscosity, based on numerical solutions of heated water wedge flows. This extended integral method, in conjunction with the Lighthill high-Prandtl-number approximation for heat transfer, can be used to compute displacement thickness, momentum thickness, wall shear stress, Nusselt number, and higher derivatives of the velocity and temperature profiles at the wall for nonsimilar boundary layers. These parameters can be computed with a hand calculator. The extended integral method is tested for the highly nonsimilar Howarth retarded flow with t_w = 104°F and t_e = 32°F; the method is accurate for this flow, except in the region near separation. (Author)

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Approximate Methods for Calculating the Properties of Heated Laminar Boundary Layers in Water

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PREFACE

The Rand Corporation, under sponsorship of the Tactical Technology Office of the Defense Advanced Research Projects Agency, has been engaged in analysis, development, and evaluation of hydrodynamic design criteria for submersible vehicles. Boundary-layer control--including shaping, pressure gradients, suction, and heating--has been the main technological basis for these investigations.

Successful boundary-layer control depends on appropriate manipulation of the laminar boundary layer to maintain stability and delay transition to turbulence. Thus, the effects of surface heating and pressure gradients on properties of the laminar boundary layer in water are central to progress in low-drag technology.

This report provides a simplified and accurate approximate method in calculating the properties of heated laminar boundary layers in water. When coupled with appropriate stability or transition criteria, it provides a systematic and efficient approach to the optimization of low-drag design.

The method presented here should simplify study of the interplay between pressure gradients and surface heating on laminar characteristics to the point where a hand calculator can be used to compute laminar-flow characteristics for the case of constant wall temperature. Additional work, still in progress, will extend this approach to the case of variable wall temperature.

This report should be useful to hydrodynamicists, to designers of submersibles, and to others interested in applying fluid mechanics to improve the performance of underwater vehicles.

Other related Rand publications are:

R-1752-ARPA/ONR, *Low-Speed Boundary-Layer Transition Workshop*, W. S. King, June 1975.

R-1789-ARPA, *Controlling the Separation of Laminar Boundary Layers in Water: Heating and Suction*, J. Aroesty and S. A. Berger, September 1975.

R-1863-ARPA, *The Effects of Wall Temperature and Suction on Laminar Boundary-Layer Stability*, W. S. King, April 1976.

R-1898-ARPA, *"e^9": Stability Theory and Boundary-Layer Transition*, S. A. Berger and J. Aroesty, February 1977.

R-1907-ARPA, *Buoyancy Cross-Flow Effects on the Boundary Layer of a Heated Horizontal Cylinder*, L. S. Yao and I. Catton, April 1976.

R-1966-ARPA, *The Buoyancy and Variable Viscosity Effects on a Water Laminar Boundary Layer Along a Heated Longitudinal Horizontal Cylinder*, L. S. Yao and I. Catton, February 1977.

R-2111-ARPA, *Entry Flow in a Heated Tube*, L. S. Yao, June 1977.

R-2164-ARPA, *The Effects of Unsteady Potential Flow on Heated Laminar Boundary Layers in Water: Flow Properties and Stability*, W. S. King, J. Aroesty, L. S. Yao, and W. Matyskiela, in process.

The authors wish to express their appreciation to C. Gazley, Jr. for reading the manuscript and offering several helpful suggestions. The authors also wish to thank W. H. Krase and L. S. Yao for their careful reviews.

SUMMARY

Thwaites[1] and others[2] have used integral methods to compute parameters of laminar boundary layers with constant fluid properties. These methods use simple correlations of universal parameters based principally on similar boundary layers. The displacement thickness, momentum thickness, and wall shear stress can be computed by these methods for nonsimilar boundary layers.

This report is concerned with the extension and application of these methods to heated water boundary layers. The viscosity of water is strongly temperature dependent, and thus variable fluid property effects are significant for heated water boundary layers. The Thwaites integral method has been extended to include variable fluid properties. Numerical solutions of heated water wedge flows are tabulated for a range of wedge angles and temperature differences. Universal parameters are correlated from these solutions for use with the integral method. The method has been tested for the nonsimilar Howarth retarded flow with $t_w = 104°F$ and $t_e = 32°F$. Errors of only 1 or 2 percent are found for displacement thickness, momentum thickness, and wall shear stress. This is a particularly stringent test because of the adverse pressure gradient in the Howarth flow, where heating causes the separation point to move 30 percent further down the surface.[3]

The curvature of the velocity profile at the surface, $f'''(0)$, depends on the viscosity gradient at the surface, which in turn is determined by the heat flux at the surface. These three additional parameters can be computed once the Nusselt number is known. The Lighthill high-Prandtl-number approximation has been extended to permit computation of the Nusselt number for boundary layers with variable fluid properties. Nusselt numbers computed with this approximate method for the heated Howarth flow are reasonably close to the exact numerical solutions for that flow, except in the region near separation.

Heated water boundary layers have been further explored with simplified shear modeling, such as Couette flow. The Couette flow model works very well at stagnation points, where the boundary layer does not grow.

CONTENTS

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NOMENCLATURE

a, b = constants

c = $\rho\mu/\rho_e \mu_e$

c_k = $\dfrac{\rho k}{(\rho k)_e} \dfrac{1}{Pr_e}$

c_p = heat capacity

f = dimensionless stream function, $f' = u/U$

F = universal parameter, $2T - 2(2 + H)\lambda$

H = universal parameter, δ^*/θ

j = 1 for axisymmetric, 0 for two-dimensional bodies

k = thermal conductivity

L = an arbitrary length

m = $(x/U)\, dU/dx$

Nu = Nusselt number, hx/k_w

Pr = Prandtl number

r = radial distance from the symmetric axis

Re_x = Reynolds number, $x\, u_e/\nu_e$

t = temperature

T = universal parameter, $\tau_w\, \theta/\mu U$

u = x component of velocity

U = velocity outside the boundary layer

v = y component of velocity

x = coordinate along the surface

y = coordinate normal to the surface

β = $2m/(m + 1)$

δ = $\left\{ 2\nu_e \int_0^x U\, dx \right\}^{\frac{1}{2}} /U$

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δ^* = displacement thickness, $\int_0^\infty (1 - u/U)\, dy$

η = similarity variable, $y/\delta(x)$

θ = momentum thickness, $\int_0^\infty \frac{u}{U} (1 - \frac{u}{U})\, dy$

$\tilde{\theta} = \theta/\delta$

$\lambda = (\theta^2/\nu_w)\, dU/dx$

μ = viscosity

ν = kinematic viscosity

ρ = density

τ = shear stress

Subscripts

e = at ambient conditions

iso = isothermal

T = of the thermal boundary layer

w = at the surface conditions

ref = at 491.69°R (32°F)

I. INTRODUCTION

The key to boundary-layer control is the maintenance of laminar velocity profiles that promote hydrodynamic stability and delay separation. For undersea applications, surface heating is a possible means of achieving this control.[2,3,4,5] Heating promotes stability through the interplay among the thermal boundary layer, the temperature-dependent viscosity of water, and the momentum balance in the crucial region near the wall.

To take full advantage of these phenomena, the surface heating distribution should be optimized, so that total heat required is minimized subject to realistic constraints. As part of a study to simplify optimization and design methods, we have developed an approximate method for determining laminar boundary-layer characteristics for cases of constant surface overheat. The extension of this approach to the case of variable surface temperature is currently under investigation at Rand.

Approximate methods are useful in optimization studies because numerical integration of the boundary-layer equations is unnecessary. If, for example, the optimal heating distribution for a given body shape is required, it may be necessary to perform hundreds of numerical calculations of the boundary-layer equations, with consequent cost in computer time. Approximate methods can reduce this cost considerably and can also yield better intuitive understanding.

For a method to be useful in predicting stability, transition, and separation, it must supply information about the velocity profiles that influence these phenomena. For example, the critical Reynolds number for stability of Tollmein-Schlichting waves has been shown to correlate with the shape factor, $H = \delta^*/\theta$, and to depend on the curvature of the velocity profile at low values of H.[6] Integral methods can be used to determine these parameters with sufficient accuracy for use in optimization studies.

We have extended Thwaites's integral method to the case of water flowing over constant temperature surfaces.* Thwaites's method depends

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*Although this method is conventionally called after Thwaites, Walz, Thwaites, and others were responsible for its development. See

on extensive correlations of parameters derived from numerically exact boundary-layer solutions, rather than a particular assumed form of velocity profile. The key simplification in the Thwaites method arises from the observation that the parameter $F = 2T - 2(2 + H)\lambda$ can be approximately expressed as $a - b\lambda$, where a and b are constants derived from a best fit to a graph of F vs λ from a computed series of different pressure gradients. In the constant property case, $a = .44$ and $b = 5.38$ are widely used values. We have found that an approximate linear relation between F and λ is still valid, but the constants a and b now depend on wall overheat and ambient temperature. As in Thwaites's original method, the momentum thickness distribution can be obtained from a simple quadrature, while shear stress, shape factors, and profile curvature can be obtained from charts or tables.

The Thwaites approach is a simple, economical, and practical scheme for the calculation of nonsimilar boundary layers and is accurate enough for most applications except in the neighborhood of laminar separation. With the extension we present here, it is possible to compute laminar boundary characteristics of water flowing over constant temperature surfaces with a hand calculator alone.

Ref. 2 for a historical survey. The method we adopt here, because of its reliance on similarity solutions, is closest to that proposed by Walz for the constant property case.

II. INTEGRAL METHOD

The conservation equations for mass, momentum, and thermal energy in a steady boundary layer are

$$\frac{\partial}{\partial x}(\rho u r^j) + \frac{\partial}{\partial y}(\rho v r^j) = 0 , \tag{1a}$$

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \rho_e U\frac{dU}{dx} + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \tag{1b}$$

$$\rho c_p\left(u\frac{\partial t}{\partial x} + v\frac{\partial t}{\partial y}\right) = \frac{\partial}{\partial y}\left(k\frac{\partial t}{\partial y}\right) , \tag{1c}$$

where $j = 1$ for axisymmetric flow and $j = 0$ for two-dimensional flow. If the density is not a strong function of temperature (as for water), the momentum equation can be integrated across the boundary layer, yielding

$$\frac{\tau_w}{\rho U^2} = \frac{d\theta}{dx} + \frac{2\theta + \delta^*}{U}\frac{dU}{dx} + \frac{j\theta}{r}\frac{dr}{dx} , \tag{2a}$$

where

$$\theta = \int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right)dy \tag{2b}$$

and

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U}\right)dy . \tag{2c}$$

The integral method developed by Thwaites[1,2] has been useful for unheated boundary layers. In this method, universal parameters correlated from a number of known solutions for similar and nonsimilar flows are applied to boundary-layer computations for any body shape. The convenient universal parameters F, H, and T, defined as

$$F = 2T - 2(2 + H)\lambda \tag{3a}$$

$$H = \delta^*/\theta \tag{3b}$$

$$T = \tau_w \theta/\mu U \quad , \tag{3c}$$

are correlated as functions of λ, defined as

$$\lambda = \frac{\theta^2}{\nu}\frac{dU}{dx} \quad . \tag{4}$$

The integrated momentum equation (2a) can be expressed as

$$\frac{1}{r^{2j}}\frac{d}{dx}\left(\frac{r^{2j}\lambda}{dU/dx}\right) = \frac{F(\lambda)}{U} \quad . \tag{5}$$

If F is linear in λ, $F = a - b\lambda$ (as shown to be approximately the case by Thwaites[1,2] for constant fluid properties), then Eq. (5) can be formally integrated, yielding

$$\theta(x) = \left(\frac{a\nu}{r^{2j}U^b}\int_0^x r^{2j}U^{b-1}\,dx\right)^{\frac{1}{2}} \quad . \tag{6}$$

For a given $U = U(x)$, θ can be obtained from this expression; λ is then obtained from its definition and H and T are then found from their correlations as functions of λ. With this procedure, $\theta(x)$, $\delta^*(x)$, and $\tau_w(x)$ can be calculated for flow over a surface of any shape, once $U(x)$ is known. It has been shown[1,2] that the Thwaites integral method accurately (within about 5 percent) determines H and skin friction for general nonsimilar laminar boundary layers with constant fluid properties, except in the region near separation.

The variable viscosity analogs to Eqs. (3) through (6) are almost identical to these constant fluid property equations. Equations (2) hold when the viscosity is temperature dependent. If the definitions in Eqs. (3)

and (4) are modified only to the extent of evaluating μ and ν at the wall temperature (replace μ and ν with μ_w and ν_w), then Eq. (5) holds for variable viscosity. The simple algebraic derivation of Eq. (5) for the variable viscosity case is nearly identical to that for the constant fluid properties case. Again, if F is assumed linear in λ, Eq. (6) will hold (with ν replaced by ν_w). In fact, for a given constant wall-to-ambient-temperature difference, it will shortly be shown that F is indeed approximately linear in λ. Thus, Eq. (6) for $\theta(x)$ holds for heated water boundary layers as well as for unheated ones, but the universal functions $F(\lambda)$, $H(\lambda)$, and $T(\lambda)$ will depend on the wall and ambient temperatures as well as on λ. These correlations can be obtained from numerical solutions for wedge flows.

For laminar boundary layer flow of a fluid with temperature-dependent fluid properties over a wedge ($j = 0$) or a cone ($j = 1$) with a constant surface-to-ambient-temperature difference, the boundary layer equations (1a through c) reduce to two ordinary differential equations, similarity equations. Thus, these wedge and cone flows are convenient for correlating the universal functions F, H, and T. The continuity equation (1a) can be eliminated by defining a stream function as[7,8]

$$r^j \rho u = \frac{\partial \psi}{\partial y} \qquad r^j \rho v = -\frac{\partial \psi}{\partial x} \ . \tag{7}$$

Now if the Mangler-Levy-Lees transformation from x and y coordinates to ξ and η coordinates is introduced,

$$d\xi = \rho_e \mu_e U (r/L)^{2j} \, dx \tag{8a}$$

$$d\eta = [\rho U/(2\xi)^{\frac{1}{2}}](r/L)^j \, dy \ , \tag{8b}$$

where L is any arbitrary length, and a dimensionless stream function, f, is defined such that

$$\psi(x,y) = (2\xi)^{\frac{1}{2}} L^j f(\xi,\eta) \ , \tag{9}$$

then the boundary layer equations (1) become[8]

$$(cf'')' + ff'' + \beta\left[\frac{\rho_e}{\rho} - (f')^2\right] = 2\xi\left(f'\frac{\partial f'}{\partial \xi} - f''\frac{\partial f}{\partial \xi}\right) \quad (10a)$$

$$(c_k g') + fg' = 2\xi\left(f'\frac{\partial g}{\partial \xi} - g'\frac{\partial f}{\partial \xi}\right) , \quad (10b)$$

where

$$\beta = \frac{2\xi}{U}\frac{dU}{d\xi} \quad (11a)$$

$$c = \frac{\rho\mu}{(\rho\mu)_e} \quad (11b)$$

$$c_k = \frac{\rho k}{(\rho k)_e}\frac{1}{Pr_e} \quad (11c)$$

$$g = \frac{T - T_0}{T_e - T_0} \quad (11d)$$

for any constant T_0, and where primes denote derivatives with respect to η. The fluid property ratios are functions of local temperature, or of g only. If β and all boundary conditions are independent of ξ, then Eqs. (10) become independent of ξ. For flow over either a wedge or a cone, the velocity outside the boundary layer is of the form[(2)] $U = U_0x^m$; for a wedge or a cone, ξ is also proportional to a power of x. Thus, for a wedge or a cone, β (from Eq. (11a)) is a constant. For an isothermal impermeable surface without slip, the boundary conditions at the surface ($\eta = 0$) are

$$f(0) = f'(0) = 0 \qquad g(0) = 0 \quad , \quad (12a)$$

if we set $T_0 = T_w$. To match the ambient temperature and velocity at the edge of the boundary layer, the boundary conditions as $\eta \to \infty$ are

$$f'(\infty) \to 1 \qquad g(\infty) \to 1 \quad . \quad (12b)$$

Therefore, for an isothermal wedge or cone, β and the boundary conditions are in fact independent of ξ, and so Eqs. (10) reduce to the similarity equations

$$(cf'')' + ff'' + \beta\left[\frac{\rho_e}{\rho} - (f')^2\right] = 0 \qquad (13a)$$

$$(c_k g')' + fg' = 0 . \qquad (13b)$$

The transformed velocity profiles for cones are the same as those for wedges; we need only correlate F, H, and T with wedge flows.

Numerical solutions of wedge flows (Eqs. (13)) with water have been obtained for various values of wedge angle and wall-to-ambient-temperature difference for an ambient temperature of 67°F (see Table 1). The water property correlations used were those of Kaups and Smith[9] (see the appendix), but density was taken as constant.

The above transformed boundary layer equations are convenient for numerical study, but the true magnitude of the heating and pressure gradient effects on the shear stress, for example, may not be obvious from the transformed shear stress, $f''(0)$. To explore the combined effects of pressure gradient and heating, and to offer further physical insight into these effects, the following simplified shear models have been developed. The simplest is to assume that heating does not change the shear in the boundary layer, the "isothermal shear model." Since τ_w is proportional to $c_w f''(0)$,

$$\frac{f''(0)}{f''_{iso}(0)} \approx c_w^{-1} . \qquad (14)$$

Another shear model, the Couette flow model, assumes that the shear stress (but not the velocity gradient) is a constant for the entire viscous boundary layer. The Couette flow model requires that some distribution of viscosity be assumed. We have assumed that $1/\mu$ is distributed linearly from $1/\mu_w$ at the wall to $1/\mu_e$ at the thermal boundary layer edge. This viscosity distribution assumption corresponds closely to a linear temperature distribution. The Couette flow model also requires the assumption that the viscous boundary layer thickness is not changed appreciably by heating; this can be shown to be the case from the top part of Table 1. This model results in

Table 1

WEDGE FLOW NUMERICAL SOLUTIONS WITH WATER AT AN AMBIENT TEMPERATURE $t_e = 67°F$

$f''(0)$, $\tilde{\theta}$, and H as functions of β and Δt

	$\Delta t = 0$, $c_w = 1.000$			$\Delta t = 10^oF$, $c_w = 0.8765$			$\Delta t = 20^oF$, $c_w = 0.7758$			$\Delta t = 30^oF$, $c_w = 0.6927$			$\Delta t = 50^oF$, $c_w = 0.5646$		
β	$f''(0)$	$\tilde{\theta}$	H	$f''(0)$	$\tilde{\theta}$	H	$f''(0)$	$\tilde{\theta}$	H	$f''(0)$	$\tilde{\theta}$	H	$f''(0)$	$\tilde{\theta}$	H
-0.18	.1293	.5665	3.298	.1611	.5607	3.164	.1947	.5543	3.047	.2299	.5474	2.943	.3038	.5327	2.769
-0.15	.2165	.5448	3.022	.2531	.5383	2.923	.2914	.5314	2.833	.3311	.5242	2.752	.4131	.5090	2.613
-0.10	.3193	.5153	2.802	.3640	.5079	2.721	.4100	.5006	2.648	.4570	.4931	2.581	.5526	.4774	2.466
-0.05	.4003	.4904	2.676	.4519	.4831	2.604	.5046	.4756	2.538	.5579	.4678	2.479	.6652	.4518	2.377
-0.025	.4361	.4795	2.630	.4908	.4722	2.561	.5464	.4645	2.498	.6026	.4567	2.441	.7152	.4406	2.344
0.0	.4696	.4696	2.592	.5272	.4620	2.525	.5857	.4543	2.464	.6445	.4464	2.409	.7622	.4303	2.315
0.025	.4980	.4631	2.558	.5615	.4526	2.493	.6226	.4448	2.435	.6840	.4368	2.382	.8064	.4207	2.291
0.05	.5311	.4514	2.529	.5941	.4438	2.466	.6577	.4359	2.409	.7215	.4280	2.358	.8484	.4117	2.269
0.10	.5870	.4354	2.481	.6549	.4277	2.421	.7232	.4197	2.366	.7915	.4116	2.318	.9268	.3954	2.234
0.20	.6867	.4082	2.411	.7631	.4003	2.355	.8398	.3922	2.304	.9161	.3842	2.259	1.066	.3678	2.182
0.50	.9277	.3502	2.297	1.025	.3422	2.248	1.121	.3341	2.204	1.216	.3261	2.166	1.401	.3103	2.101
1.00	1.233	.2923	2.217	1.354	.2844	2.174	1.475	.2766	2.136	1.593	.2689	2.103	1.821	.2540	2.048

$f'''(0)$ and $Nu/Re^{\frac{1}{2}}$ as functions of β and Δt

	$\Delta t = 10^oF$		$\Delta t = 20^oF$		$\Delta t = 30^oF$		$\Delta t = 50^oF$	
β	$f'''(0)$	$Nu/Re^{\frac{1}{2}}$	$f'''(0)$	$Nu/Re^{\frac{1}{2}}$	$f'''(0)$	$Nu/Re^{\frac{1}{2}}$	$f'''(0)$	$Nu/Re^{\frac{1}{2}}$
-0.18	0.1907	0.4860	0.1983	0.5017	0.2023	0.5163	0.2023	0.5432
-0.15	0.1459	0.5359	0.1382	0.5489	0.1272	0.5613	0.09638	0.5844
-0.10	0.07470	0.5894	0.04500	0.6010	0.01141	0.6121	-0.06525	0.6330
-0.05	0.005474	0.6291	-0.04420	0.6401	[illegible]	0.6507	-0.2169	0.6706
-0.025	-0.02865	0.6463	-0.08782	0.657[illegible]	[illegible]	0.6676	-0.2902	0.6871
0.0	-0.06256	0.6623	-0.1310	0.673[illegible]	[illegible]	0.6833	-0.3622	0.7026
0.025	-0.09626	0.6774	-0.1737	0.6881	[illegible]	0.6982	-0.4329	0.7173
0.05	-0.1298	0.6918	-0.2161	0.7023	-0.3077	0.7124	-0.5029	0.7313
0.10	-0.1963	0.7189	-0.2996	0.7293	-0.4089	0.7393	-0.6401	0.7580
0.20	-0.3280	0.7690	-0.4640	0.7794	-0.6068	0.7892	-0.9067	0.8075
0.50	-0.7152	0.9110	-0.9417	0.9214	-1.177	0.9313	-1.665	0.9496
1.00	-1.347	1.201	-1.709	1.212	-2.083	1.223	-2.854	1.243

$$\frac{f''(0)}{f''_{iso}(0)} \simeq \frac{c_w^{-1}}{1 + \frac{1}{2}(\delta_T/\delta)(c_w^{-1} - 1)} \quad , \tag{15}$$

where the ratio of boundary-layer thicknesses can be approximated by

$$\frac{\delta_T}{\delta} \simeq \overline{Pr}^{-1/3} \quad . \tag{16}$$

These simplified shear models suggest that heating may affect f"(0) by approximately the same factor for all pressure gradients; this f"(0) ratio is shown in Table 2 for wedge flows with various values of β, and in fact, this ratio is fairly constant for a given Δt. Note that the Couette flow model is very good at $\beta = 1$, stagnation point flow, where the boundary layer does not grow (see Table 2). The isothermal shear model corresponds closely to $\beta = -0.1$ (see Table 2), for which it can be seen from Table 1 that the shear stress (proportional to $c_w f''(0)$) is nearly the same for all heating levels.

Returning now to our original analysis, we note that Eqs. (3) for F, H, and T can be expressed in terms of the transformed variables. With the density taken as constant, for a wedge transformation (8b) becomes $\eta = y/\delta(x)$, where $\delta(x) = \sqrt{2 - \beta}\; Re_x^{-\frac{1}{2}}$. Equations (3) become

$$F = 2(1 - \beta)\tilde{\theta}^2 c_w^{-1} \tag{17a}$$

$$H = \delta^*/\theta \tag{17b}$$

$$T = \tilde{\theta} f''(0) \tag{17c}$$

for isothermal wedge flows, where $\tilde{\theta} = \theta/\delta$. For wedge flows, λ can be found from the following boundary condition at the wall

$$\frac{\partial}{\partial y}\left(\mu\,\frac{\partial u}{\partial y}\right)_w = \rho_w\, U\,\frac{dU}{dx} = \lambda\,\frac{\mu_w}{\theta^2}\,U \quad , \tag{18}$$

from which

Table 2

COMPARISON OF SIMPLIFIED SHEAR MODELS WITH NUMERICAL SOLUTIONS OF f"(0) FOR HEATED WATER BOUNDARY LAYERS OVER WEDGES ($t_e = 67^oF$)

		$f''(0)/f''_{iso}(0)$			
β	$f''_{iso}(0)$	$\Delta t = 10^oF$	$\Delta t = 20^oF$	$\Delta t = 30^oF$	$\Delta t = 50^oF$
- .18	.1286	1.246	1.506	1.778	2.349
- .15	.2164	1.169	1.347	1.530	1.909
- .10	.3193	1.140	1.284	1.432	1.731
- .05	.4003	1.129	1.260	1.394	1.662
0.00	.4696	1.123	1.247	1.372	1.623
.10	.5870	1.116	1.232	1.348	1.579
.20	.6867	1.112	1.223	1.334	1.553
.50	.9277	1.104	1.208	1.311	1.511
1.00	1.233	1.098	1.196	1.292	1.477
Isothermal shear model		1.141	1.289	1.444	1.771
Couette flow model		1.100	1.195	1.286	1.453

$$\lambda = \frac{\theta^2}{U}\left[\frac{1}{\mu_w}\left(\frac{d\mu}{dy}\right)_w\left(\frac{\partial u}{\partial y}\right)_w + \left(\frac{\partial^2 u}{\partial y^2}\right)_w\right] \tag{19a}$$

$$\lambda = \tilde{\theta}^2\left[\frac{c'_w}{c_w} f''(0) + f'''(0)\right] \tag{19b}$$

or

$$\lambda = -\beta\tilde{\theta}^2 c_w^{-1} \quad . \tag{19c}$$

F, H, and T have beer calculated from Eqs. (17) using Table 1, and are plotted in Figs. 1, 2, and 3 as functions of λ, computed from Eq. (19c). Actually, as indicated, the F and H curves are plotted as functions of $\bar{\lambda} = c_w\lambda$ instead of λ, and $\bar{F} = c_w F$ is plotted instead of F. This removes c_w from F and λ (see Eqs. (17a) and (19c)) and makes the resulting curves nearly parallel and closer together. Fortunately, as is seen in Fig. 1, F is nearly linear in λ for a given Δt, so that the extension of Thwaites's method is straightforward. The form $F = a - b\lambda$ can be used, but the constants a and b now depend on temperature. As shown in Figs. 1, 2, and 3,

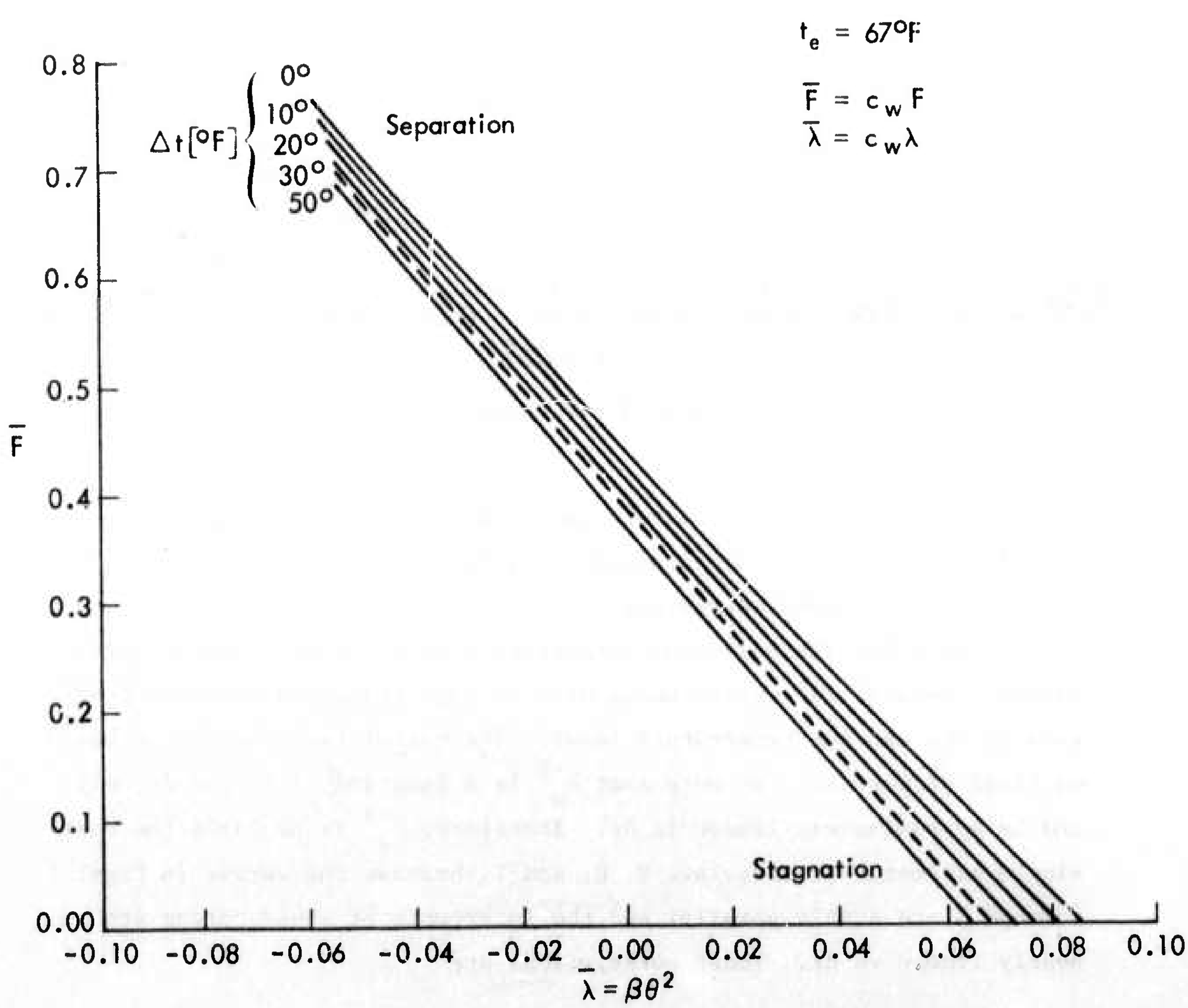

Fig. 1 — $\bar{F}$ versus $\bar{\lambda}$

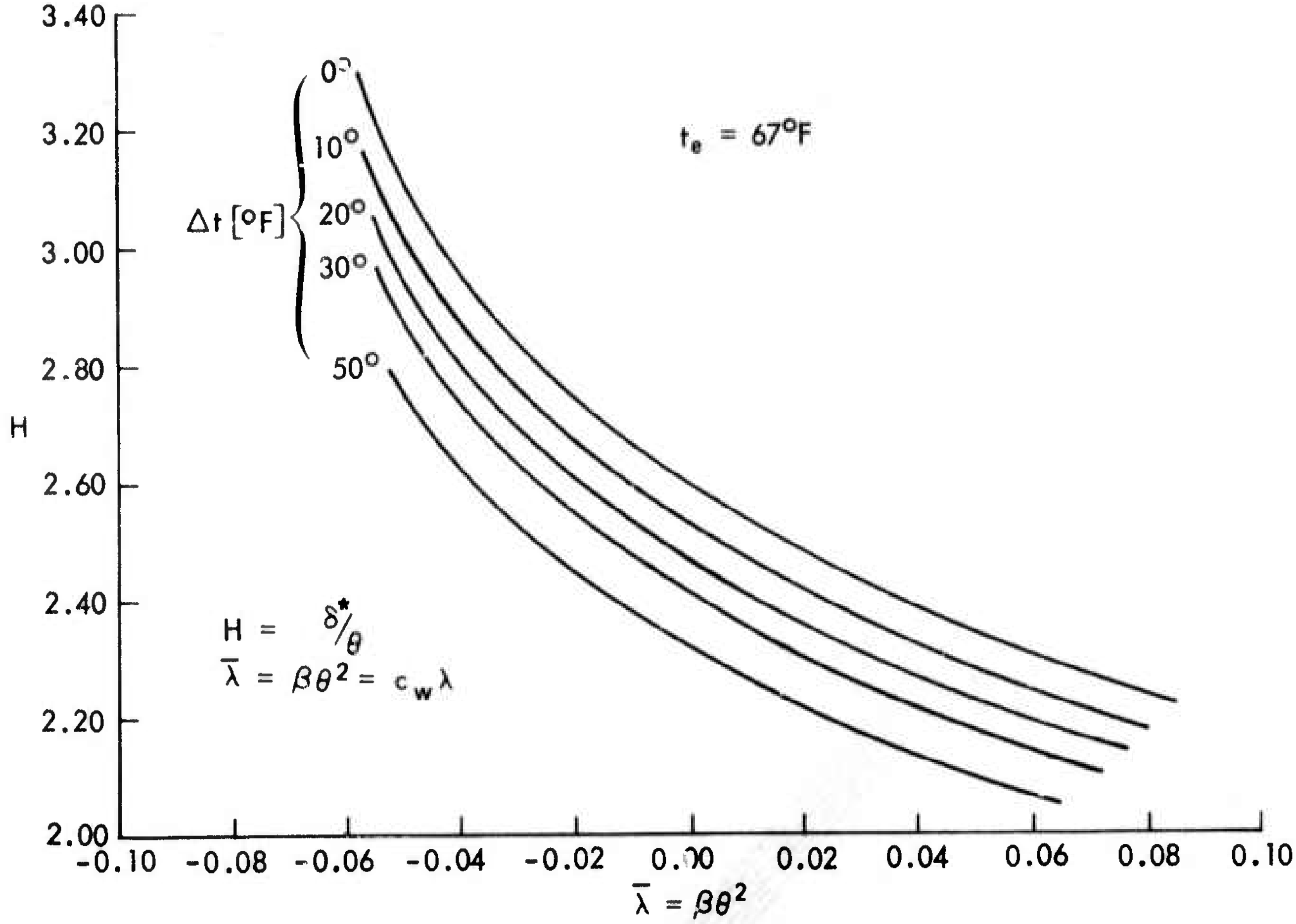

Fig. 2 — H versus $\overline{\lambda}$

F and H decrease with temperature and T increases with temperature. This is because θ decreases with temperature, but not as fast as does δ*, and f"(0) increases with temperature.

Correlating the universal parameters with Δt is not totally satisfactory, because the curves would also be expected to have some dependence on the ambient temperature level. The correlations should be based on fluid properties. We note that c_w^{-1} is a function of t_e and Δt only and is approximately linear in Δt. Therefore, c_w^{-1} is probably the best single parameter to correlate F, H, and T, because the curves in Figs. 1 through 3 are nearly parallel and the intercepts of these curves are nearly linear in Δt. These correlations are

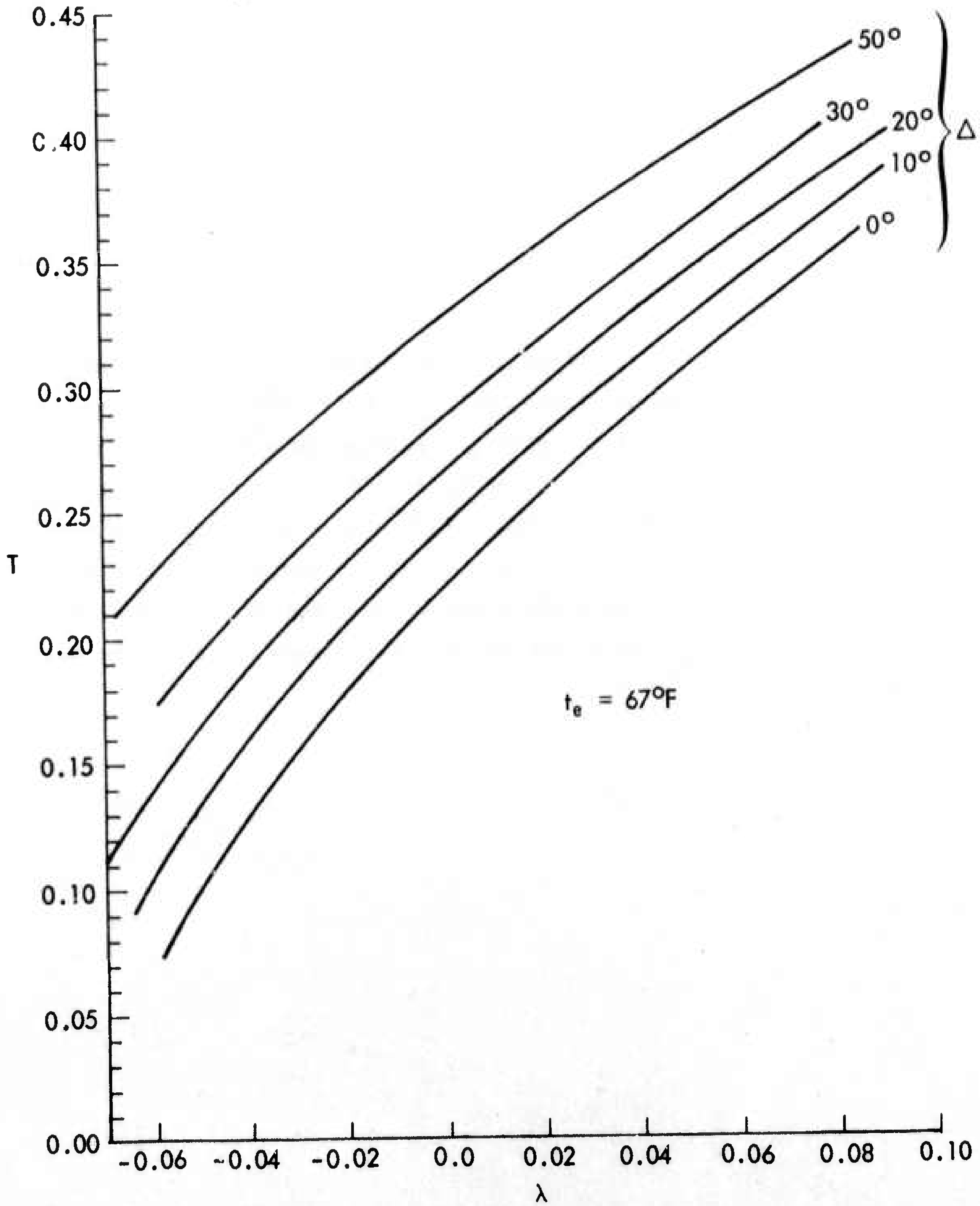

Fig. 3—T versus λ

$$F = a - b\lambda \tag{20a}$$

$$a = [0.441 - 0.0953(c_w^{-1} - 1)]c_w^{-1}$$

$$b = 5.38 + 0.415(c_w^{-1} - 1)$$

$$H = H_0(\lambda) - 0.410(c_w^{-1} - 1) \tag{20b}$$

$$T = T_0(\lambda) + (0.149 - 0.730\lambda)(c_w^{-1} - 1) \ , \tag{20c}$$

where H_0 and T_0 are the isothermal H and T values. These correlations are based on wedge flow solutions with $t_e = 67°F$, and they have been checked at several other ambient temperatures. Perhaps an additional correlating parameter, such as Pr_e or c_w', is needed in addition to c_w^{-1}. At $t_e = 47°F$ and $\Delta t = 20°F$, a, b, and H are predicted by the above correlations within 1 percent; for this case, T is predicted within 4 percent except for values of λ near separation where T can be in error by as much as 12 percent. However, at $t_e = 32°F$, the above correlations are poor, as might be expected.

III. HEAT TRANSFER

Water has a high Prandtl number: in the temperature range of interest for underwater vehicles, the Prandtl number of water ranges from 4 to 13. The Lighthill high-Prandtl-number approximation for heat transfer[10,7] works well in this range; in fact, for a flat plate with zero pressure gradient at a Prandtl number of one, the high-Prandtl-number approximation is in error by only 2 percent. For constant fluid properties, the high-Prandtl-number approximation for heat transfer[10,7] is

$$Nu_x = 0.5384\, Pr^{1/3} \frac{x}{\nu} \sqrt{\frac{\tau_w}{\rho}} \left\{ \int_0^x \sqrt{\frac{\tau_w}{\rho}} \frac{dx}{\nu} \right\}^{-1/3} . \qquad (21)$$

For this same formula to hold for water, with its temperature-dependent viscosity, a reference temperature must be used. As demonstrated in the following section, this reference temperature approach meets with acceptable but limited success.

The Nusselt number is a dimensionless surface heat flux or a dimensionless temperature gradient at the surface,

$$Nu_x = \frac{q_w x}{k_w (t_w - t_e)}$$

where

$$q_w = k_w \left.\frac{\partial t}{\partial \eta}\right|_w \left.\frac{\partial \eta}{\partial y}\right|_w .$$

When the Nusselt number has been determined, the viscosity ratio gradient at the wall can be computed from the expression

$$c'_w = \frac{1}{\mu_e} \left.\frac{d\mu}{d\eta}\right|_{\eta=0} = \frac{1}{\mu_e} \left.\frac{d\mu}{dt}\right|_{t=t_w} \left.\frac{dt}{d\eta}\right|_{\eta=0} . \qquad (22)$$

It is sometimes useful or necessary to have expressions for higher derivatives of the velocity at the wall (e.g., see comments in Sec. I). By

taking the limit of Eq. (10a) as $\eta \to 0$, $f'''(0)$ can be expressed as

$$f'''(0) = -[\beta + c_w' f''(0)]/c_w \ . \tag{23}$$

Higher derivatives of f and t at the wall can be similarly found such that, e.g., series expansions for the temperature and velocity profiles near the wall can be obtained.

Thus θ, δ^*, $f''(0)$, Nu_x, c_w', $f'''(0)$, and other parameters can all be found for nonsimilar heated water boundary layers by simple integrals and a table or chart. The procedure, for an arbitrary pressure gradient of the nonsimilar type, is to integrate Eq. (6) to obtain the momentum thickness θ, then use Eq. (4) and Figs. 1, 2, and 3 to obtain the shape factor H, and the shear parameter T. With T and θ known, then $\tau_w(x) = \mu_w UT(x)/\theta(x)$ is computed, and the integration in the high-Prandtl-number heat-transfer formula (Eq. (21)) can be performed.

IV. EXAMPLE: HOWARTH'S LINEARLY RETARDED FLOW

The methods described in the previous sections have been applied to a demanding test case with an adverse pressure gradient and strongly varying fluid properties. An exact numerical solution[2] of Howarth's retarded flow [$U = u_o(1 - x/8L)$] for water with $t_w = 104°F$ and $t_e = 32°F$ has been compared with the integral method solution to the same problem. The correlations (Eqs. (20)) cannot be extrapolated to this low an ambient temperature, so numerical solutions for water wedge flows were made with $t_w = 104°F$ and $t_e = 32°F$. $\overline{F}$, H, and T are plotted as functions of $\overline{\lambda}$ in Fig. 4. Again, $\overline{F}(\overline{\lambda})$ is a nearly straight line with a = 0.904 and b = 6.05. The comparison between the exact and integral solutions (see Table 3) shows excellent agreement for θ and good agreement for δ^* and $f''(0)$. Note that $\beta = -.18$ corresponds to $\lambda \simeq .13$ for wedge flows, and our computations beyond this point (x/L greater than 0.8 for Howarth's flow) are extrapolations. Howarth's flow, having an adverse pressure gradient, is strongly affected by variable fluid properties. With the above temperature boundary conditions, separation occurs at x/L = 1.246, while in the constant fluid properties case ($t_w = t_e$) separation occurs at x/L = 0.958 (the "8" in the U equation was arbitrarily chosen to make x/L near one at separation). This shift of the separation point is a dramatic variable-fluid property effect. Thus, this flow is a good check case for the integral method.

The high Prandtl number approximation was applied to this flow, with the same surface and ambient temperatures, using the numerical solutions for $f''(0)$ (see Table 3), the average Prandtl number, and the surface values of τ and ν. The Nusselt numbers from this approximate formula are compared with exact numerical solutions[3] in Table 4. The difference is only about 4 percent for most of the plate, but the difference is much larger near separation.

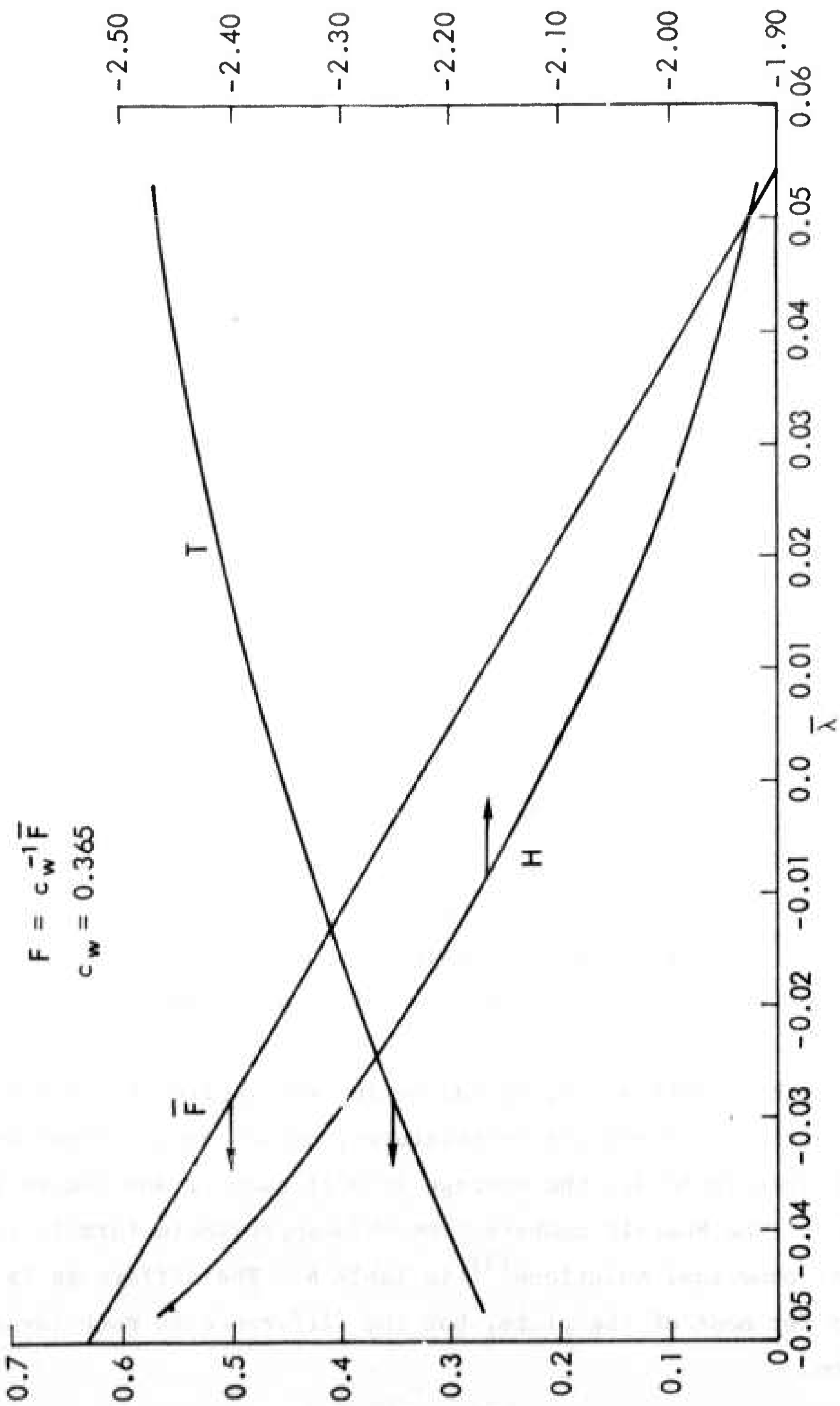

Fig. 4—$\bar{F}$, H, and T versus $\bar{\lambda}$ for $t_w = 104°F$ and $t_e = 32°F$

Table 3

COMPARISON OF THE INTEGRAL METHOD TO NUMERICAL SOLUTIONS FOR HOWARTH'S RETARDED FLOW (WATER WITH t_w = 104°F AND t_e = 32°F)

x/L	λ	β	Numerical Solutions[a]			Integral Method		
			$\theta Re_x^{1/2}/x$	δ^*/θ	$f''(0)$	$\theta Re_x^{1/2}/x$	δ^*/θ	$f''(0)$
0.0	0.	0.	.573	2.13	1.117	.574	2.12	1.113
.2	-.0247	0.0526	.592	2.17	1.022	.593	2.17	1.022
.4	-.0544	-.111	.613	2.22	.914	.614	2.24	.911
.6	-.0900	-.176	.637	2.28	.789	.637	2.33	.772
.8	-.133	-.250	.663	2.36	.641	.661	2.48	.594
1.0	-.186	-.333	.693	2.50	.454	.689		
1.1	-.216	-.379	.710	2.60	.334	.704		
1.2	-.250	-.429	.728	2.78	.170	.719		
1.246	-.267	-.452	.737	3.01	.016	.726		

[a]Reference (3).

Table 4

COMPARISON OF THE HIGH-PRANDTL-NUMBER APPROXIMATION TO NUMERICAL SOLUTIONS FOR HOWARTH'S RETARDED FLOW (WATER WITH t_w = 104°F AND t_e = 32°F)

x/L	Numerical Solution[a]	High-Prandtl-Number Approximation
	$Nu_x/Re_x^{1/2}$	$Nu_x/Re_x^{1/2}$
0.0	0.923	0.908
.2	0.890	0.869
.4	0.853	0.820
.6	0.807	0.763
.8	0.750	0.689
1.0	0.669	0.582
1.1	0.610	0.501
1.2	0.511	0.359
1.246	0.357	0.111

[a]Reference (3).

V. CONCLUSIONS

The classical isothermal constant-properties integral method of Walz and Thwaites has been extended to heated water boundary layers, where temperature-dependent viscosity is important. Correlations of the universal parameters F, H, and T as simple functions of c_w^{-1} and λ allow computation of θ, δ^*, and $f''(0)$ for heated water boundary layers with any surface shape and with most temperature levels of interest for underwater vehicles.

The method has been tested for the Howarth retarded flow with $t_w = 104°F$ and $t_e = 32°F$. With water, this flow is very temperature dependent; heating in this flow causes the separation point to move 30 percent further down the plate(3); the flow is highly nonsimilar even without heating. The value of θ was calculated by the integral method to within 1 percent nearly to the separation point. The values of δ^* and $f''(0)$, both nearly exact at x = 0, had each attained an error of only 2 percent at a position halfway to the separation point.

The method shares several advantages and shortcomings with the original methods upon which it is based. Its advantages are that the momentum thickness and heat transfer can be calculated accurately, easily, and economically for a wide variety of pressure gradients, both favorable and adverse. The shortcomings are that it may not represent H and T adequately, particularly in regions of adverse gradients, and thus the location of laminar separation may not be determined accurately.

The high-Prandtl-number approximation has also been found to be accurate in computing the heat transfer in heated water boundary layers, except in the region near separation.

Despite these limitations, this method extends a proven and reliable approach to the computation of laminar boundary layers in water.

Appendix

FLUID PROPERTY CORRELATIONS

The water fluid properties correlations used in the numerical work for this report are those of Kaups and Smith.[9] The density is taken as constant, so only the viscosity ratio, thermal conductivity ratio, and Prandtl number correlations are needed. With T expressed in degrees Rankine, and T_{ref} = 491.69°R (32°F), these correlations are

$$\mu/\mu_{ref} = 1/[35.15539 - 106.9718715\ (T/T_{ref}) + 107.7720376\ (T/T_{ref})^2 - 40.5953074\ (T/T_{ref})^3 + 5.6391948\ (T/T_{ref})^4]$$

$$k/k_{ref} = -1.940589 + 5.2220185\ (T/T_{ref}) - 2.693322\ (T/T_{ref})^2 + 0.4176167\ (T/T_{ref})^3$$

$$Pr = 13.66/[73.376906 - 208.7474538\ (T/T_{ref}) + 197.7604676\ (T/T_{ref})^2 - 68.8626186\ (T/T_{ref})^3 + 7.4779458\ (T/T_{ref})^4].$$

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